Lec 2

I n vectors and the vector space $\mathbb{R}^n$.

An n-vector is an ordered n-tuple

$$(x_1 \ x_2 \ x_3 \ldots x_n)$$

$x_1, x_2 \ldots$ are called the components of the n-vector.

n is the dimension

Norm:

$$\|(x_1 \ldots x_n)\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

Addition:

$$(x_1, x_2 \ldots, x_n) + (y_1, y_2, \ldots, y_n) =$$

$$(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

Scaler multiplication:

$$\lambda (x_1, x_2, \ldots, x_n) = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n)$$

Dot product, Inner product:

$$(x_1, x_2, \ldots, x_n) \cdot (y_1, y_2, \ldots, y_n) =$$

$$x_1 y_1 + x_2 y_2 + \cdots + x_n y_n .$$

Null vector:

$$0 = (0, 0, \ldots, 0)$$

$$x = (x_1, x_2, \ldots, x_n)$$

Properties: $\quad y = (y_1, y_2, \ldots, y_n)$

(i) $x + y = y + x$.

(ii) $x + 0 = 0 + x = x$

(iii) $(x + y) + z = x + (y + z)$

(iv) $\lambda(x + y) = \lambda x + \lambda y$.

(v) $(\lambda\mu)x = \lambda(\mu x) = \mu(\lambda x)$.

(vi) $(\lambda + \mu)x = \lambda x + \mu x$.

(vii) $\|\lambda x\| = |\lambda| \, \|x\|$.

Properties of dot product:

(i) $x \cdot y = y \cdot x$

(ii) $x \cdot (y + z) = x \cdot y + x \cdot z$.

(iii) $(\lambda x) \cdot y = x \cdot (\lambda y) = \lambda (x \cdot y)$.

(iv) $x \cdot x = \|x\|^2$.

(v) $x \cdot 0 = 0$.

(vi) $\|x\|^2 = 0$ iff $x = 0$.

Inequalities:

(i) $|x \cdot y| \leq \|x\| \, \|y\|$ (Cauchy Schwarz)

(ii) $\|x + y\| \leq \|x\| + \|y\|$. (triangle).

II for any x, y it follows that

$$-1 \leq \frac{x \cdot y}{\|x\| \, \|y\|} \leq 1$$

[From Cauchy Schwarz]

We define

$$\cos\theta = \frac{x \cdot y}{\|x\| \, \|y\|}.$$

θ is the angle between the vectors x and y.

When $\theta = 90^\circ$ ie $\cos\theta = 0$ we have $x \cdot y = 0$.

"Thus x and y are perpendicular to each other when $x \cdot y = 0$."

III Define

$$e_i = (0, 0, \ldots 0, 1, 0, 0, \ldots, 0)$$

i^{th} spot.

$$i = 1, 2, \ldots, n.$$

We can write

$$x = \sum_{i=1}^{n} x_i e_i$$

$$= x_1 e_1 + x_2 e_2 + \cdots + x_n e_n.$$

Also
$$e_i \cdot e_j = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

Example 1 :

Let x and y be two perpendicular vectors in $\mathbb{R}^n$. Prove the pythagorus theorem

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 .$$

Solution:

$$\|x+y\|^2 = (x+y)\cdot(x+y)$$
$$= x\cdot(x+y) + y\cdot(x+y)$$
$$= x\cdot x + x\cdot y + y\cdot x + y\cdot y .$$
$$= \|x\|^2 + 2\, x\cdot y + \|y\|^2 .$$

If $x\cdot y = 0$ we have

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 .$$

Example 2

Let x and y be two vectors in $\mathbb{R}^n$ and let λ be any scalar. Prove that

$$||x+\lambda y||^2 + ||x-\lambda y||^2 = 2\left(||x||^2 + \lambda^2 ||y||^2\right)$$

Example 3

Under what condition on x, y would we have

$$|x \cdot y| = ||x|| \, ||y||$$ ← (Cauchy Schwarz Equality)

Sol^n.

When $x = \lambda y$, for some scalar λ, we have

$$x \cdot y = \lambda y \cdot y = \lambda \|y\|^2 .$$

Hence

$$|x \cdot y| = |\lambda| \, \|y\|^2 .$$

$$\|x\| \, \|y\| = \|\lambda y\| \, \|y\|$$
$$= |\lambda| \, \|y\| \, \|y\|$$
$$= |\lambda| \, \|y\|^2 .$$

It follows that

$$x = \lambda y \implies |x \cdot y| = \|x\| \, \|y\| .$$

Conversely, recall that

$$x \cdot y = \|x\| \, \|y\| \cos\theta .$$

$$\implies |x \cdot y| = \|x\| \, \|y\| \, |\cos\theta|$$

If $|x \cdot y| = \|x\| \, \|y\|$, we have

$$|\cos\theta| = 1 \implies \theta = 0^\circ \text{ or } 180^\circ .$$

Thus $x = \lambda y$ for some scalar λ.

IV Linear Combination

Let $x_1, x_2, \ldots, x_m$ be a set of n-vectors in $\mathbb{R}^n$, then a linear combination is an expression of the form

$$c_1 x_1 + c_2 x_2 + \cdots + c_m x_m .$$

where $c_1, c_2, \ldots, c_m$ are scalars.

Ex: $\mathbb{R}^4$

$$x_1 = (1 \quad 2 \quad 3 \quad 4)$$

$$x_2 = (2 \quad 1 \quad 4 \quad 1)$$

$$x_3 = (6 \quad 0 \quad 2 \quad 2)$$

$$2x_1 + x_2 + 3x_3 =$$

$$(2 \quad 4 \quad 6 \quad 8) + (2 \quad 1 \quad 4 \quad 1) + (18 \quad 0 \quad 6 \quad 6)$$

$$= (22 \quad 5 \quad 16 \quad 15)$$

V Linear independence

Let $x_1, \ldots, x_m$ be a set of n-vectors in $\mathbb{R}^n$. The set is said to be linearly independent iff

$$c_1 x_1 + \cdots + c_m x_m = 0$$

$$\Rightarrow c_1 = c_2 = \cdots = c_m = 0.$$

Note: When $c_1 = c_2 = \cdots = c_m = 0$, the linear combination

$$c_1 x_1 + c_2 x_2 + \cdots + c_m x_m$$

is called "trivial linear combination".

"A set of vectors are called linearly independent iff only linear combination which sums up to zero is the trivial linear combination".

VI Linear dependence

The set of vectors

$x_1, \cdots, x_m$

would be linearly dependent if there exist at least one non-trivial linear combination.

$$c_1 x_1 + c_2 x_2 + \cdots + c_m x_m .$$

where $(c_1, c_2 \cdots, c_m) \neq 0$.

and such that

$$c_1 x_1 + \cdots + c_m x_m = 0 .$$

Without any loss of generality assume that $c_1 \neq 0$, we conclude that.

$$\boxed{x_1 = -\frac{c_2}{c_1} x_2 - \frac{c_3}{c_1} x_3 - \cdots - \frac{c_m}{c_1} x_m}$$

Thus we have the following:

A set of vectors $x_1, \ldots, x_m$ in $\mathbb{R}^n$ would be linearly dependent if any vector can be written as a linear combination of the other vectors.

Example:

Are the vectors

$x_1 = (2 \quad 1 \quad 1 \quad 0)$

$x_2 = (0 \quad 2 \quad 0 \quad 1)$

$x_3 = (1 \quad 1 \quad 0 \quad 2)$

$x_4 = (0 \quad 2 \quad 1 \quad 1)$

linearly dependent or linearly independent ? !

Sol^{n}

write down a l.c. and set it equal to 0.

$$c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4 = 0$$

$$\Rightarrow (2c_1 \quad c_1 \quad c_1 \quad 0) +$$
$$(0 \quad 2c_2 \quad 0 \quad c_2) +$$
$$(c_3 \quad c_3 \quad 0 \quad 2c_3) +$$
$$(0 \quad 2c_4 \quad c_4 \quad c_4) = (0 \quad 0 \quad 0 \quad 0)$$

$\Rightarrow 2c_1 + c_3 = 0$ $\qquad$ $c_1 + c_4 = 0$

$c_1 + 2c_2 + c_3 + 2c_4 = 0$ $\qquad$ $c_2 + 2c_3 + c_4 = 0$

I $c_1 = -c_4$

II $c_3 = -2c_1 = 2c_4$

III $c_1 + 2c_2 + c_3 + 2c_4 = 0$

$\Rightarrow -c_4 + 2c_2 + 2c_4 + 2c_4 = 0$

$\Rightarrow 3c_4 + 2c_2 = 0$

$\Rightarrow c_2 = -\frac{3}{2}c_4$

IV. $c_2 + 2c_3 + c_4 = 0$

$-\frac{3}{2}c_4 + 4c_4 + c_4 = 0$

$(5 - \frac{3}{2})c_4 = 0 \Rightarrow \boxed{c_4 = 0}$

$7/2\, c_4 = 0$

$\Rightarrow \boxed{c_1 = c_2 = c_3 = c_4 = 0}$

Hence the vectors are linearly independent

Example:

Are the vectors

$$x_1 = (2 \quad 1 \quad 1 \quad 0)$$

$$x_2 = (0 \quad 2 \quad 0 \quad 1)$$

$$x_3 = (1 \quad 1 \quad 0 \quad 2)$$

$$x_4 = (3 \quad 4 \quad 1 \quad 3)$$

linearly dependent or independent.

Sol$^{\underline{n}}$

Write down a l.c. and set it equal to 0.

$$c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4 = 0.$$

$$\Rightarrow (2c_1 \quad c_1 \quad c_1 \quad 0) +$$
$$(0 \quad 2c_2 \quad 0 \quad c_2) + \quad = (0 \quad 0 \quad 0 \quad 0)$$
$$(c_3 \quad c_3 \quad 0 \quad 2c_3) +$$
$$(3c_4 \quad 4c_4 \quad c_4 \quad 3c_4)$$

2.16

$$\Rightarrow 2c_1 + c_3 + 3c_4 = 0$$

$$c_1 + 2c_2 + c_3 + 4c_4 = 0$$

$$c_1 + c_4 = 0$$

$$c_2 + 2c_3 + 3c_4 = 0$$

$$\Rightarrow c_1 = -c_4 .$$

$$2c_1 + c_3 + 3c_4 = 0$$

$$\Rightarrow -2c_4 + c_3 + 3c_4 = 0$$

$$\Rightarrow c_4 + c_3 = 0 \Rightarrow c_3 = -c_4 .$$

$$c_1 + 2c_2 + c_3 + 4c_4 = 0$$

$$\Rightarrow -c_4 + 2c_2 - c_4 + 4c_4 = 0$$

$$\Rightarrow 2c_2 + 2c_4 = 0 \Rightarrow c_2 = -c_4 .$$

$$c_2 + 2c_3 + 3c_4 = 0$$

$$\Rightarrow -c_4 - 2c_4 + 3c_4 = 0$$

$$\Rightarrow 0 = 0$$

Taking

$c_4 = \lambda$ an arbitrary real number, we get

$c_1 = -\lambda$

$c_2 = -\lambda$

$c_3 = -\lambda$

Conclusion:

$c_1 = 0$, $c_2 = 0$, $c_3 = 0$, $c_4 = 0$ is not ~~the~~ only solution

Hence vectors are linearly dependent.